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Properties of Close Packing of Filaments in Yarn

Abstract

In this study a hypothetical model of the close cross-sectional structure of yarn and some peculiarities of close packing are analysed. The aim of our research is to propose new methods for calculating the packing indices of close-packed yarn. As the basis of one method, a mathematical description of each filament (fibre) position in the yarn cross-section was assumed. This method was suggested for a yarn consisting of a finite number of ring layers. Another method was proposed for the case of an infinite number of filaments or ring layers in yarn cross-section. By analysing the simplest basic element of the cross-section, it was shown that in such a case the packing fraction equals 0.906. The results of the current paper and the data published in other scientific works are in good agreement.

Key words: *close packing, yarn structure, yarn cross-section, packing model.*

Introduction

Yarn packing has practical significance for a number of the characteristics of various yarns and fabrics. A great many attempts to produce packing models of yarns have thus been made. The structure of real yarns is quite disordered, and in many models this is replaced by an idealised structure. One of the basic forms of packing models is close packing, in which the filaments (fibres) fit into a hexagonal pattern.

As a theoretical object of yarn structural morphology, close packing has been analysed over the last few decades. For example, close packing was discussed in papers published by Gracie [1], Hearle & co-authors [2,3], Iyer Balakrishna & Phatarfod [4], Neckář &Ježek [5], Zemlekov & Popov [6], Perepelkin [7], and Morris, Merkin & Renell [8].

Literature Survey

A classic example of close packing of filaments gives a yarn with a hexagonal outline, in which all filaments are touching as shown in Figure 1. The model of the yarn is as a rule composed of equal, non-compressible, circular filaments. A main peculiarity of close packing is that the voids between the filaments are minimal. As mentioned earlier, the filaments are arranged in a hexagonal pattern. Therefore cross-sectional layers take the shape of a six-sided figure. The model is approximated to an equilateral figure because usually the first layer is a single fibre forming a core. The distance between the axis of yarn and the centres of each layer (radius of layer) varies, as indicated by Hearle & co-authors [3], from the value $2(i-1)r_f$ at the corner of hexagon to the value $3^{1/2}(i-1)r_f$ at the middle of the side of the hexagon, where i is the current number of the hexagonal layer (i=1,...,t); t is the number of hexagonal layers in the yarn, and F_r is the filament radius.

The number of filaments in layer i, as indicated by Hearle and co-authors [3], is:

$$a_i = K(i-1) \quad (1)$$

where K is the coefficient of proportionality. K=6 and $i \geq 2$ in equation (1).
An analogous method was suggested by Iyer Balakrishna & Phatarfod [4]:

$$B=6A \quad (2)$$

where:
B - the number of filaments in the hexagonal layer,
A - the current number of the hexagonal layer (the central filament is not counted as a layer).

A simple method may be used to calculate the total number of close-packed filaments [4]:

$$n=1+3A(A+1) \quad (3)$$

According to Hearle & co-authors [3], the indices of the close-packed model of the yarn cross-section which is approximated to an equilateral hexagon are presented in Table 1, where i is the current number of the hexagonal layer (i=1,...,t); R_i is the radius of the hexagonal layer i; n_i is the number of filaments in the hexagonal layer i; t is the number of hexagonal layers in the yarn, and n is the total number of filaments in the yarn consisting of the hexagonal layers t. The same quantity of n was indicated by Gracie [1]. It should be noted that index R_8 for a corner filament is above index R_9 for a filament in the middle of the

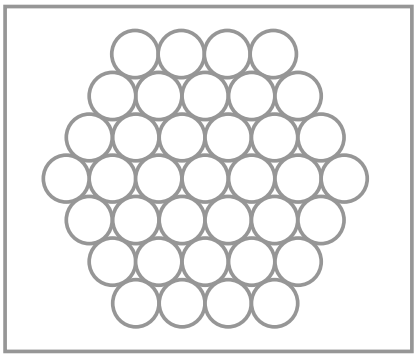

Figure 1. *Cross-section of hexagonal close-packet yarn.*

Table 1. Indices of close-packed model of yarn cross-section, which is approximated to equilateral hexagon.

Indices of layer				Indices of yarn	
i	R_i at corner	R_i at middle of side	n_i	t	n
1	0	0	1	1	1
2	$2r_f$	$1.73r_f$	6	2	7
3	$4r_f$	$3.46r_f$	12	3	19
4	$6r_f$	$5.20r_f$	18	4	37
5	$8r_f$	$6.93r_f$	24	5	61
6	$10r_f$	$8.66r_f$	30	6	91
7	$12r_f$	$10.4r_f$	36	7	127
8	$14r_f$	$12.1r_f$	42	8	169
9	$16r_f$	$13.9r_f$	48	9	217

side. Zemlekov & Popov [6] proposed a model of close packing the notions of layer and intermediate layer. This suggestion means that a cross-section of yarn is approximated to a circle. Unfortunately, Zemlekov & Popov [6] used only a graphic method to describe the position of the filaments in each concentric ring layer.

In order to define the properties of packing filaments in a yarn, an index of packing fraction is usually used. The packing fraction of the yarn may be defined as the ratio of the volume of constituent filaments to the volume of the yarn:

$$\Phi=V_f/V_y \quad (4)$$

where:
V_f - the total volume of filaments in the yarn;
V_y - the volume of the yarn.

This definition corresponds with the explanation mentioned by Iyer Balakrishna & Phatarfod [4], Hearle & co-authors [3], Neckář [9], etc. The volume V_f and the volume V_y for yarn, which consists of filaments of the same initial length, may be expressed in this way:

$$V_f=m_f/\rho \quad (5)$$

and

$$V_y=m_y/\sigma \quad (6)$$

where:
m_f - the total mass of filaments in volume V_f;
m_y - is the mass of yarn in volume V_y;
ρ - the filament density;
σ - the yarn density.

As the mass m_f and the mass m_y are equal values ($m_f=m_y$) in a yarn of fixed filament length, then equation (4) is

$$\Phi=\rho/\sigma \quad (7)$$

Equation (7) was used to express the packing fraction, for example, in the works published by Binkevičius [10] and Neckář [11]. Gracie [1] has given the packing fraction as the ratio of the area of the constituent circles of yarn cross-section to the area of the circumscribing circle of the cross-section. According to this definition,

$$\Phi=A_f/A_y \quad (8)$$

where:
A_f - the total cross-sectional area of filaments in the yarn;
A_y - the cross-sectional area of the yarn.

It has been reported [3] that the experimental packing fractions range from just over 0.3 in some spun yarns up to nearly 0.9 for a highly twisted nylon filament yarn. The model of close packing is suitable for yarns of the highest packing fraction.

The Principle of the Method and the Results of Mathematical Description of the Filament Position

One aim of the current paper is to propose a new numerical method for the mathematical description of each filament position in close-packed yarn. The yarn is assumed to have a circular cross-section, as shown in Figure 2. The filaments are close-packed together in this cross-section; therefore the voids between the filaments are minimal, as typical for a close-packed hexagon. Only one-sixth of the yarn cross-section need be investigated, because the arrangement of filaments in this segment is similar to the arrangement in other segments. It is possible to express the position of any filament in respect of the axis of the yarn (central filament) by calculating the distance between this axis and the axis of the filament cross-section. This distance is equal to the length of vector **r** (Figure 3). The length of vector $\boldsymbol{r}$ depends on the length of vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$:

$$\boldsymbol{r}=\boldsymbol{r}_1+\boldsymbol{r}_2 \quad (9)$$

For the equilateral hexagonal model, an angle between vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ equals $\pi/3$. According to the cosine theorem, the length of vector $\boldsymbol{r}$ is:

$$r=\{r_1^2+r_2^2-2r_1r_2\cos(2\pi/3)\}^{1/2} \quad (10)$$
$$=(r_1^2+r_2^2+r_1r_2)^{1/2}$$

where:
r_1 - the length of vector $\boldsymbol{r}_1$;
r_2 - the length of vector $\boldsymbol{r}_2$.

The values of r_1 or r_2 depend on the filament radius r_f and the current number of the layer i:

$$r_1=2(i-1)r_f \quad (11)$$

and

$$r_2=2(i-1)r_f \quad (12)$$

In order to obtain results comparable with the data published by Zemlekov & Popov [6], the position of filaments in a cross-section of yarn was investigated for 12 concentric ring layers. These results are presented in Tables 2 and 3. According to the length r (calculated in the proposed way), it was established in which cross-sectional ring layer each filament lies. For

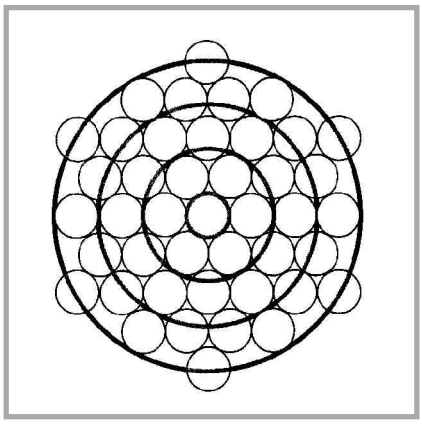

Figure 2. View of cross-section of close-packed yarn after approximation to circle; first layer - 1 filament, second layer - 6 filaments, third layer - 12 filaments, fourth layer - 24 filaments.

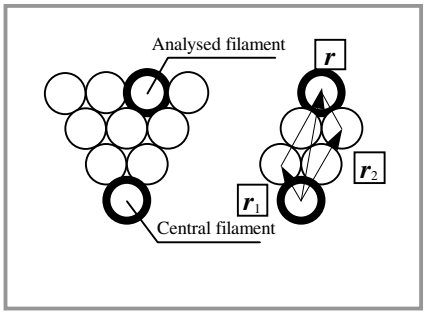

Figure 3. Scheme describing filament position in respect of central filament for close-packed model.

example, if the length r varies between $13r_f$ and $15r_f$, the filament lies in the layer, which current number i=8. These results of length r are indicated by putting a mark * in Table 2. There are 7 filaments which satisfy this requirement, in Table 2, but there are no such filaments in Table 3. Naturally, as each layer consists of 6 segments, there are 42 such filaments in the above-mentioned ring layer. In addition 6 filaments lie in each layer (except the first layer). So in all, 48 filaments enter the eighth (i=8) layer.

Table 4 summarises the results of such calculations for each layer. Definitions for the parameters are as follows:

- i - the current number of ring layer;
- R_i - the radius of current ring layer;
- n_{li} - the number of filaments which entirely lie in the current ring layer;
- n_{ii} - the number of filaments in the current ring's intermediate layer;
- n_i - the total number of filaments in the current ring layer;
- t - the number of ring layers in the yarn;
- n - the total number of filaments in the yarn consisting of ring layers t.

It should be observed that there is absolute agreement between the theoretically calculated values of index n and the results established graphically by Zemlekov & Popov [6]. When approximating a close-packed model to a circle, the total number of filaments does not increase in regular order with the increase in the number of ring layers. The additional filaments are placed in ring layers for which i=4 and i=11.

Results of Packing Indices for a Yarn Consisting of a Finite Number of Cross-Sectional Ring Layers

Another stage of this study was to calculate a packing fraction for each ring layer Φ_i and for the whole yarn cross-section Φ. In accordance with equation (8) these indices are, respectively:

$$\Phi_i = A_{fi}/A_{yi} \quad (13)$$

and

$$\Phi=(A_{f1}+...+A_{fi}+...+A_{ft})/(A_{y1}+...+A_{yi}+...+A_{yt}) = \Sigma A_{fi}/\Sigma A_{yi} \quad (14)$$

where:

- A_{fi} - the total sum of the cross-sectional area of all filaments in the current layer i;
- A_{yi} - the cross-sectional area of the current layer i.

It is obvious that the approximation of a hexagonal pattern to a circle does not permit any homogeneous packing fraction in different layers (see Table 5). The index Φ_i fluctuates between 0.750 and 1.050. The latter value even exceeds the highest possible boundary of the yarn's packing fraction. An evident increase in packing fraction was calculated for the above-mentioned layers i=4 and i=11. This growth was obtained at the expense of a decrease in packing fraction in neighbouring layers.

The average value of index Φ for a yarn consisting of various number of ring layers (t=1,...,12) equals 0.878. The packing fraction Φ, as can be seen from Table 6, is fairly stable if the number of layers t>3. The deviation of index Φ from the average value does not exceeds 2.6% if t=4,...,12. With the increment of the total number of layers, the packing fraction Φ becomes more stable because the influence of different packing in separate layers is eliminated. It is thus possible to think that the value Φ=0.898 (for t=12) very well defines the packing intensity of yarn consisting of a great number of filaments.

Table 2. *Calculated values of length r for filaments in intermediate layers.*

r		For r_1					
		$2r_f$	$4r_f$	$6r_f$	$8r_f$	$10r_f$	$12r_f$
For r_2	$2r_f$	$3.46r_f$	$5.29r_f$	$7.21r_f$	$9.17r_f$	$11.14r_f$	$13.11r_f$*
	$4r_f$	$5.29r_f$	$6.93r_f$	$8.72r_f$	$10.58r_f$	$12.49r_f$	$14.42r_f$*
	$6r_f$	$7.21r_f$	$8.72r_f$	$10.39r_f$	$12.17r_f$	$14.00r_f$*	$15.87r_f$
	$8r_f$	$9.17r_f$	$10.58r_f$	$12.17r_f$	$13.86r_f$*	$15.62r_f$	$17.44r_f$
	$10r_f$	$11.14r_f$	$12.49r_f$	$14.00r_f$*	$15.62r_f$	$17.32r_f$	$19.08r_f$
	$12r_f$	$13.11r_f$*	$14.42r_f$*	$15.87r_f$	$17.44r_f$	$19.08r_f$	$20.78r_f$
	$14r_f$	$15.10r_f$	$16.37r_f$	$17.78r_f$	$19.29r_f$	$20.88r_f$	$22.54r_f$
	$16r_f$	$17.09r_f$	$18.33r_f$	$19.70r_f$	$21.17r_f$	$22.72r_f$	-
	$18r_f$	$19.08r_f$	$20.30r_f$	$21.63r_f$	$23.07r_f$	-	-
	$20r_f$	$21.07r_f$	$22.27r_f$	$23.58r_f$	-	-	-
	$22r_f$	$23.07r_f$	-	-	-	-	-
	$24r_f$	-	-	-	-	-	-

Table 3. *Calculated values of length r for filaments in intermediate layers.*

r		For r_1					
		$14r_f$	$16r_f$	$18r_f$	$20r_f$	$22r_f$	$24r_f$
For r_2	$2r_f$	$15.10r_f$	$17.09r_f$	$19.08r_f$	$21.07r_f$	$23.07r_f$	-
	$4r_f$	$16.37r_f$	$18.33r_f$	$20.30r_f$	$22.27r_f$	-	-
	$6r_f$	$17.78r_f$	$19.07r_f$	$21.63r_f$	$23.58r_f$	-	-
	$8r_f$	$19.29r_f$	$21.17r_f$	$23.07r_f$	-	-	-
	$10r_f$	$20.88r_f$	$22.72r_f$	-	-	-	-
	$12r_f$	$22.54r_f$	-	-	-	-	-

Table 4. *Indices of close-packed model of yarn cross-section, which is approximated to circle.*

Indices of layer					Indices of yarn	
i	R_i	n_{li}	n_{ii}	n_i	t	n
1	0	1	0	1	1	1
2	$2r_f$	6	0	6	2	7
3	$4r_f$	6	6	12	3	19
4	$6r_f$	6	18	24	4	43
5	$8r_f$	6	24	30	5	73
6	$10r_f$	6	30	36	6	109
7	$12r_f$	6	36	42	7	151
8	$14r_f$	6	42	48	8	199
9	$16r_f$	6	48	54	9	253
10	$18r_f$	6	54	60	10	313
11	$20r_f$	6	78	84	11	397
12	$22r_f$	6	72	78	12	475

Table 5. Results of packing fraction calculation for separate cross-sectional layers of close-packed yarn cross-section, approximated to circle.

i	A_{yi}	A_{fi}	Φ_i
1	πr_f^2	πr_f^2	1.000
2	$8\pi r_f^2$	$6\pi r_f^2$	0.750
3	$16\pi r_f^2$	$12\pi r_f^2$	0.750
4	$24\pi r_f^2$	$24\pi r_f^2$	1.000
5	$32\pi r_f^2$	$30\pi r_f^2$	0.938
6	$40\pi r_f^2$	$36\pi r_f^2$	0.900
7	$48\pi r_f^2$	$42\pi r_f^2$	0.875
8	$56\pi r_f^2$	$48\pi r_f^2$	0.857
9	$64\pi r_f^2$	$54\pi r_f^2$	0.844
10	$72\pi r_f^2$	$60\pi r_f^2$	0.833
11	$80\pi r_f^2$	$84\pi r_f^2$	1.050
12	$88\pi r_f^2$	$78\pi r_f^2$	0.886

Table 6. The results of packing fraction calculation for close-packed yarn cross-section, approximated to circle.

i	ΣA_{yi}	ΣA_{fi}	Φ
1	πr_f^2	πr_f^2	1.000
2	$9\pi r_f^2$	$7\pi r_f^2$	0.778
3	$25\pi r_f^2$	$19\pi r_f^2$	0.760
4	$49\pi r_f^2$	$43\pi r_f^2$	0.878
5	$81\pi r_f^2$	$73\pi r_f^2$	0.901
6	$121\pi r_f^2$	$109\pi r_f^2$	0.901
7	$169\pi r_f^2$	$151\pi r_f^2$	0.893
8	$225\pi r_f^2$	$199\pi r_f^2$	0.884
9	$289\pi r_f^2$	$253\pi r_f^2$	0.875
10	$361\pi r_f^2$	$313\pi r_f^2$	0.867
11	$441\pi r_f^2$	$397\pi r_f^2$	0.900
12	$529\pi r_f^2$	$475\pi r_f^2$	0.898

Counting Method of Packing Fraction in the Case of an Infi-nite Number of Filaments or Ring Layers

In addition, one more method for index Φ counting was suggested. This method predicts the value of index F if the total number of filaments in the yarn n or number of ring layers t increases infinitely.

The main point of this method is the use of the simplest geometrical element, which is typical of close packing of filaments in the cross-section of yarn. The filaments of the close-packed model may be arranged in comb-type cells of a yarn cross-section, as shown in Figure 4. The triangle and the sector of a filament in Figure 5 illustrate an element which can be considered as a component of a comb-type cell of the yarn cross-section. Each comb-type cell can be formed in two-dimensional space using eleven reflection operations with a right-angle triangle, which serves as the simplest element, and alternately changes the axis of reflection. The triangle is repeated by reflection as if by a mirror, which turns at a 30° angle around the centre of the comb-type cell. For example, during the first reflection, the axis of reflection coincides with a cathetus of the right-angle triangle. For the second time, the axis of reflection coincides with the hypotenuse. A subsequent order of reflection is repeated until the comb-type cell is formed.

There is one void and a sector of one filament in the element shown in Figure 4. For such an element the cross-sectional area of the segment of a filament is:

$$A_{fe}=\pi r_f^2/12 \qquad (15)$$

The area of element of the yarn is:

$$A_{ye}=r_f^2/(2x3^{1/2}) \qquad (16)$$

This approach gives the packing fraction of the element of a yarn:

$$\Phi_e=A_{fe}/A_{ye}=\pi/(2x3^{1/2})=0.906 \qquad (17)$$

Some Comparisons of Count

It should be observed that other investigators obtained their results theoretically or proposed them without calculation. The packing fraction values in order of increase are given in Table 7. For example, Neckář [11], and Neckář & Ježek [5] used a formula to count the packing fraction of a hexagonal yarn structure in which the concept of a gap between filaments was introduced. According to these authors, the packing fraction is:

$$\mu=\{\pi/(2x3^{1/2})\}(1+a/d)^{-2} \qquad (18)$$

where:
a - the gap between filaments;
d - the filament diameter.

A similar possibility for reducing the packing density of fibres for a hexagonal packing model was mentioned by Morris, Merkin & Rennell [8]. They suggested reducing the value of close packing density (0.9) by filling only a proportion of the spaces. It is interesting to note that for the case without a gap we have *a*=0. For this kind of yarn

$$\mu=\pi/(2x3^{1/2}).$$

This value is analogous to index Φ_e as calculated in the current paper.

Table 7 lists the values of packing fraction for a yarn also consisting of an infinite number of filaments. Examples of such results are those proposed by Gracie [1], Iyer Ba-

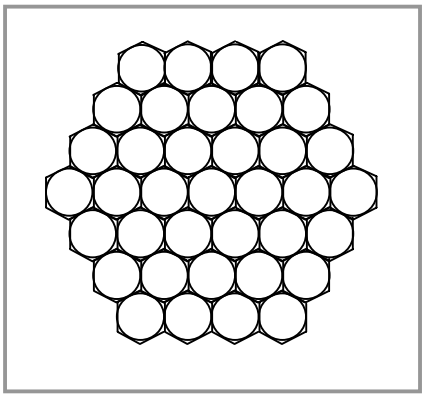

Figure 4. Close-packed filaments in comb-type cells of yarn cross-section.

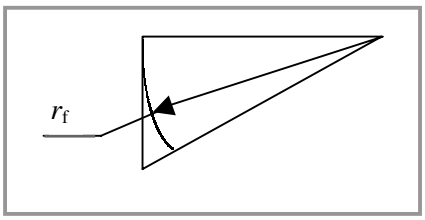

Figure 5. One-sector component of comb-type cell of close-packed yarn cross-section.

Table 7. The values of the indices of packing fraction of close-packed yarn according to literary data.

Packing fraction	Authors
0.9	Morris, Merkin & Rennell, 1999 [8]
0.906	Hearle & Merchant, 1963 [2]
0.9069	Iyer Balakrishna & Phatarfod, 1965 [4]
0.907	Perepelkin, 1997 [7]
0.907	Neckář & Ježek, 1985 [5]
0.9078	Gracie, 1960 [1]
0.91	Hearle & co-authors, 1969 [3]
0.92	Hearle in Chou & Ko (eds), 1991 [12]

lakrishna & Phatarfod [4]. Although the circumstance of an infinite quantity of filaments is not indicated by the other authors mentioned in Table 7, the published values of packing fraction differ from Φ_e=0.906 by no more than 1.5%. The counting results of the packing fraction for *a* large number of filaments n or ring layers t are similar to the limiting value of the packing fraction as well. For example, the difference between index Φ=0.898 for t=12 (Table 6) and Φ_e=0.906 equals 0.9%.

Concluding Remarks

The proposed description of the close packing of filaments in a yarn provides a new numerical method for expressing the position of filaments in respect of the yarn axis. The addition of vectors is the essence of the method. The estimation allowed the calculation of the vector ***r*** for each filament. According to the length of vector ***r***, it is possible to determine in which cross-sectional ring layer a given filament lies. Such an approach allows calculation of the packing indices for close-packed yarn cross-sections, which is approximated to a circle consisting of a finite number of ring layers. An essential part of another method is to use comb-type cells in the yarn cross-section. The method uses the simplest basic element of close-packed structure. This approach serves as a guide for calculating in a simple manner the packing fraction in the situation when the number of filaments in a yarn is large.

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